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Relation between 'spin' and 'orbital' angular momenta in classical fields†

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Abstract. For a scalar field interacting with its associated electromagnetic field the 'orbital' and 'spin' angular momenta are shown to be proportional. An appropriate divergence added to the Lagrangian ensures that the 'orbital' part is zero in the zero-momentum frame.

Current interest in classical fields, especially with regard to extended objects, leads naturally to considerations of the formulation of the angular momentum. It is well known that the division of the total angular momentum into an 'orbital' part and a 'spin' part is not unique in classical field theories. Indeed, as Belinfante (1940) has shown, two different Lagrangians generating the same field equations can give a different splitting of the total angular momentum. In addition, for a gauge-invariant system the splitting depends in general on the choice of gauge. These ambiguities are highlighted by the fact that the so called 'orbital' angular momentum is not necessarily zero in the zero-momentum frame.

In this paper we show that for Lagrangians involving the interaction of a scalar field with its associated electromagnetic field the 'orbital' and 'spin' angular momenta are actually proportional. This then allows us to make the 'orbital' part zero in the zero-momentum frame by adding an appropriate divergence to the Lagrangian.

Consider the Lagrangian ($c = 1$, $x_4 = \mathrm{i}t$)

$$\mathscr{L} = \mathscr{L}_{\mathrm{EM}} + \bar{\mathscr{L}} + \mathscr{L}'$$

where the electromagnetic part is given by

$$\mathscr{L}_{\mathrm{EM}} = -\tfrac{1}{4}F^2_{\mu\nu}$$

and

$$\bar{\mathscr{L}} = \bar{\mathscr{L}}(u, v, w, z^i, \psi^i)$$

is a function of the Lorentz invariants

$$u = A_{\alpha,\alpha} \qquad v = \tfrac{1}{2}(\psi^i_{,\alpha})^2 \qquad w = A^2_\alpha \qquad z^i = A_\alpha\psi^i_{,\alpha}$$

where ψ^i ($i = 1, 2$) is a charged scalar field. The only requirement on the functional form of $\bar{\mathscr{L}}$ is that sufficiently localised regular static solutions exist, which then define the rest-frame of the system.

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Also included in the Lagrangian is the pure divergence

$$\mathscr{L}' = K(u^2 - A_{\mu,\nu}A_{\nu,\mu}) = K\, \partial_\mu (A_\mu A_{\nu,\nu} - A_\nu A_{\mu,\nu})$$

where K is an arbitrary constant.

The total angular momentum can be written as

$$J_k = L_k + S_k$$

where the 'orbital' part

$$L_k = \epsilon_{klm} \int x_l T_{m\lambda}\, \mathrm{d}\sigma_\lambda, \tag{1}$$

$T_{\mu\nu}$ being the canonical energy–momentum tensor

$$T_{\mu\nu} = \delta_{\mu\nu}\mathscr{L} - A_{\alpha,\mu}\frac{\partial \mathscr{L}}{\partial A_{\alpha,\nu}} - \psi^i_{,\mu}\frac{\partial \mathscr{L}}{\partial \psi^i_{,\nu}}, \tag{2}$$

and the 'spin' part for this system

$$S_k = \epsilon_{klm} \int A_l \frac{\partial \mathscr{L}}{\partial A_{m,\lambda}}\, \mathrm{d}\sigma_\lambda. \tag{3}$$

Using

$$\frac{\partial \mathscr{L}}{\partial A_{\alpha,\nu}} = (1-2K)A_{\nu,\alpha} - A_{\alpha,\nu} + \frac{\partial \mathscr{L}}{\partial u}\delta_{\alpha\nu} \tag{4}$$

$$\frac{\partial \mathscr{L}}{\partial \psi^i_{,\alpha}} = \frac{\partial \mathscr{L}}{\partial v}\psi^i_{,\alpha} + \frac{\partial \mathscr{L}}{\partial z^i}A_\alpha \tag{5}$$

one can write, for static solutions, on the hypersurface $x_4 = \text{constant}$ with $A_4 = i\phi$,

$$T_{m4} = -\mathrm{i}\left((1-2K)A_{n,m}\phi_{,n} + \frac{\partial \mathscr{L}}{\partial u}\phi_{,m} + \frac{\partial \mathscr{L}}{\partial z^i}\phi\psi^i_{,m}\right)$$

so that (1) and (3) become ($\mathrm{d}\sigma_4 = -\mathrm{i}\,\mathrm{d}^3x$)

$$L_k = -\epsilon_{klm}\int x_l\left((1-2K)A_{n,m}\phi_{,n} + \frac{\partial \mathscr{L}}{\partial u}\phi_{,m} + \frac{\partial \mathscr{L}}{\partial z^i}\phi\psi^i_{,m}\right)\mathrm{d}^3x \tag{6}$$

$$S_k = (1-2K)\epsilon_{klm}\int A_l\phi_{,m}\,\mathrm{d}^3x. \tag{7}$$

To show that (6) and (7) are proportional we will need the field equations for A_μ. Making use of (4) we find, for the static case,

$$(1-2K)A_{n,mn} - A_{m,nn} + \left(\frac{\partial \mathscr{L}}{\partial u}\right)_{,m} - \frac{\partial \mathscr{L}}{\partial z^i}\psi^i_{,m} - 2\frac{\partial \mathscr{L}}{\partial w}A_m = 0 \tag{8}$$

$$\phi_{,nn} + 2(\partial \mathscr{L}/\partial w)\phi = 0. \tag{9}$$

The explicit dependence of (8) on K is only apparent; it is cancelled by the contribution from $(\partial \mathscr{L}'/\partial u)_{,m}$. Adding A_m times (9) to ϕ times (8) and integrating gives

$$0 = \epsilon_{klm}\int x_l\left\{\phi\left[(1-2K)A_{n,mn} - A_{m,nn} + \left(\frac{\partial \mathscr{L}}{\partial u}\right)_{,m} - \frac{\partial \mathscr{L}}{\partial z^i}\psi^i_{,m}\right] + A_m\phi_{,nn}\right\}\mathrm{d}^3x. \tag{10}$$

Thus, using (10) we can express (6) as

$$L_k = -\epsilon_{klm} \int x_l\left[(1-2K)(\phi A_{n,m})_{,n} + \left(\phi \frac{\partial \mathscr{L}}{\partial u}\right)_{,m} + (A_m\phi_{,nn} - \phi A_{m,nn})\right] \mathrm{d}^3x. \tag{11}$$

Consider now, in turn, each of the three terms in (11). The term multiplied by $(1-2K)$ can be written as

$$\begin{aligned} -\epsilon_{klm} \int x_l(\phi A_{n,m})_{,n}\, \mathrm{d}^3x &= -\epsilon_{klm} \int [(x_l\phi A_{n,m})_{,n} - (\phi A_l)_{,m} + \phi_{,m}A_l]\, \mathrm{d}^3x \\ &= -\epsilon_{klm} \int [(x_l\phi A_{n,m})_{,n} - (\phi A_l)_{,m}]\, \mathrm{d}^3x - \frac{1}{(1-2K)} S_k \end{aligned} \tag{12}$$

where use has been made of (7).

The second term in (11) is

$$-\epsilon_{klm} \int x_l\left(\phi \frac{\partial \mathscr{L}}{\partial u}\right)_{,m} \mathrm{d}^3x = -\epsilon_{klm} \int \left(x_l\phi \frac{\partial \mathscr{L}}{\partial u}\right)_{,m} \mathrm{d}^3x, \tag{13}$$

while the third term is

$$\begin{aligned} -\epsilon_{klm} &\int x_l(A_m\phi_{,nn} - \phi A_{m,nn})\, \mathrm{d}^3x \\ &= -\epsilon_{klm} \int \{[x_l(A_m\phi_{,n} - \phi A_{m,n})]_{,n} + (\phi A_m)_{,l} - 2\phi_{,l}A_m\}\, \mathrm{d}^3x \\ &= -\epsilon_{klm} \int \{[x_l(A_m\phi_{,n} - \phi A_{m,n})]_{,n} + (\phi A_m)_{,l}\}\, \mathrm{d}^3x - \frac{2}{(1-2K)} S_k \end{aligned} \tag{14}$$

where, again, we have used (7).

Substituting (12), (13) and (14) into the expression (6) for L_k gives

$$\begin{aligned} L_k = -\left(\frac{2}{1-2K} + 1\right) S_k &- \epsilon_{klm} \int \left[\phi\left(x_l \frac{\partial \mathscr{L}}{\partial u} - 2(1-K)A_l\right)\right]_{,m} \mathrm{d}^3x \\ &- \epsilon_{klm} \int \{x_l[(1-2K)\phi A_{n,m} - \phi A_{m,n} + A_m\phi_{,n}]\}_{,n}\, \mathrm{d}^3x. \end{aligned} \tag{15}$$

The integrals appearing in (15) can all be converted to surface integrals at spatial infinity and will therefore vanish if the fields A_μ and ψ' go to zero fast enough. For example, if, as $r \to \infty$, $\phi \sim 1/r$, $\boldsymbol{A}$ goes to zero faster than $1/r$ and $\partial\mathscr{L}/\partial u$ goes to zero faster than $1/r^2$, then the integrals vanish. Thus for time-independent, sufficiently localised solutions

$$L_k = -\left(\frac{2}{1-2K} + 1\right) S_k. \tag{16}$$

We note that the total angular momentum

$$J_k = L_k + S_k = -\frac{2}{1-2K} S_k = -2\epsilon_{klm} \int A_l\phi_{,m}\, \mathrm{d}^3x \tag{17}$$

is independent of K as required.

Finally, it is evident from (16) that, for non-zero S_k, the choice $K = \frac{3}{2}$ makes $L_k = 0$ and S_k then represents the total intrinsic angular momentum of the system.

Reference

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